

# Some applications of the parameterized Picard-Vessiot theory

Claude Mitschi

**ABSTRACT.** This expository article, intended for a special volume in memory of Andrey Bolibrukh, describes some applications of the parameterized Picard-Vessiot theory. This Galois theory for parameterized linear differential equations was Cassidy and Singer's contribution to an earlier volume dedicated to Bolibrukh. The main results we present here were obtained in joint work with Michael Singer, for families of ordinary differential equations with parameterized regular singularities. They include 'parametric' versions of the Schlesinger theorem and of the weak Riemann-Hilbert problem as well as an algebraic characterization of a special type of monodromy evolving deformations, illustrated by the classical Darboux-Halphen equation. Some of these results were recently applied by different authors to solve the inverse problem in parameterized Picard-Vessiot theory, and were also generalized to irregular singularities. We sketch some of these results by other authors. The paper includes a brief history of the Darboux-Halphen equation as well as an appendix about differentially closed fields.

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## 1. Parameterized Picard-Vessiot theory

The classical Picard-Vessiot theory, or differential Galois theory, PV-theory for short, associates with any linear differential system

$$(1) \quad \partial Y = AY$$

where the entries of the square matrix  $A$  belong to a differential field  $k$  of characteristic zero with derivation  $\partial$  and algebraically closed field of constants, a so-called Picard-Vessiot extension of  $k$ . This is a differential field extension of  $k$  generated by the entries of a fundamental solution, it has no new constants and its derivation is given by (1). Picard-Vessiot extensions are unique up to differential  $k$ -isomorphisms, and their group of differential  $k$ -automorphisms is called the Picard-Vessiot group, or differential Galois group. It is a linear algebraic group, which reflects many properties of the equation, such as its solvability, reducibility, existence of algebraic solutions etc.

In the special volume [17] dedicated to Andrey Bolibrukh, Cassidy and Singer developed a *parameterized Picard-Vessiot theory*, PPV-theory for short, based on seminal work by Cassidy, Kolchin and Landesman. In PPV-theory, the differential base-field  $k$  is endowed with a set of commuting derivations  $\Delta = \{\partial_0, \partial_1 \dots \partial_m\}$ . As in PV-theory, one wants to associate with a (square) differential system

$$(2) \quad \partial_0 Y = AY$$

with coefficients in  $k$ , a unique *parameterized Picard-Vessiot extension*, that is, a  $\Delta$ -differential field extension of  $k$  generated by the entries of a fundamental solution of (2) (generated as a field extension by these entries and their  $\Delta$ -derivatives at any order) with no new  $\partial_0$ -constants. The *parameterized Picard-Vessiot group* of a PPV-extension is its group of  $\Delta$ -differential  $k$ -automorphisms, with the usual expected properties such as a parameterized version of ‘‘Galois correspondence’’. The following example (cf. [17] p.118) shows that some assumptions are needed to meet these requirements.

EXAMPLE 1.1. Consider the scalar differential equation

$$(3) \quad \frac{dy}{dx} = \frac{t}{x}y$$

For fixed  $t \in \mathbb{C}$  we can apply classical PV-theory over the differential fields  $\mathbb{C}(x)$  or  $\overline{\mathbb{Q}}(x)$  for instance. An easy calculation on the solution  $x^t$  shows that the PV-group of (3) over  $\mathbb{C}(x)$  (resp.  $\overline{\mathbb{Q}}(x)$ ) is  $\mathbb{C}^*$  (resp.  $\overline{\mathbb{Q}}^*$ ) if  $t \notin \mathbb{Q}$ , a cyclic subgroup (of roots of unit) else.

If we now consider (3) as a parameterized family over the differential field  $k = \mathbb{C}(x, t)$  of rational functions in  $x$  and  $t$  with derivations  $\{\frac{d}{dx}, \frac{d}{dt}\}$ , its PPV-extension is

$$K = \mathbb{C}(x, t, x^t, \log x).$$

Let us show that the corresponding PPV-group is

$$G = \mathbb{C}^*$$

and that  $\log x$  is an element of  $K$  invariant by  $G$ , whereas the subfield  $K^G$  of  $K$  of elements left invariant by  $G$  should be the base-field  $k$  if  $G$  satisfied Galois correspondence. Since an element  $\sigma \in G$  is determined by  $\sigma(x^t)$  and  $\sigma(\log x)$  and commutes with both derivations, it is of the form

$$\sigma(x^t) = a_\sigma x^t, \quad \sigma(\log x) = \log x + c_\sigma$$

where  $c_\sigma$  is the logarithmic derivative of  $a_\sigma$ , and  $a_\sigma \in \mathbb{C}^*$ ,  $c_\sigma \in \mathbb{C}$  only depend on  $t$ . An easy calculation shows that

$$G = \{a \in \mathbb{C}(t)^*, a''a - a'^2 = 0\} = \mathbb{C}^*$$

where  $a', a''$  are the first and second derivatives with respect to  $t$ , and that  $G = \mathbb{C}^*$  since the  $a$  are rational functions of  $t$ , which in particular implies that  $c_\sigma = 0$  for all  $\sigma \in G$ , hence  $\sigma(\log x) = \log x$  for all  $\sigma \in G$ .

To have  $K^G = k$  in Example 1.1, the group  $G$  needs to be larger, hence contain non-constant elements. If one assumes the field  $k_0 = k^{\partial_0}$  of  $\partial_0$ -constants to be *differentially closed* (see the Appendix) then Cassidy and Singer ([17], p.116) proved that for any equation (2) there is a unique PPV-extension of  $k$  and that its PPV-group is a linear *differential algebraic group* defined over  $k_0$ , that is, a subgroup of  $\mathrm{GL}(n, k_0)$  defined by differential polynomial equations, in other words, closed in the Kolchin topology, whose elementary closed sets are the zero sets of  $\{\partial_1, \dots, \partial_m\}$ -differential polynomials. For more facts about differential algebraic groups we refer to the work of Cassidy [16], who first introduced these objects, and to [26], [13]. A Galois correspondence now holds between closed differential subgroups of the PPV-group and intermediate  $\Delta$ -differential extensions of  $k$  in the PPV-extension.

Note that since  $k_0$  is assumed to be differentially closed, it is in particular algebraically closed, and usual PV-theory holds for Equation (2). The PPV-group, which is Kolchin-closed in  $\mathrm{GL}(n, k_0)$ , is not closed in general in the (weaker) Zariski-topology and its Zariski-closure is precisely the PV-group.

In what follows we only consider families of differential equations whose coefficients are complex analytic functions, depending analytically on complex parameters. In the parametric case we first need to clarify the notion of regular singular points.

## 2. Parameterized singular points

Consider a family of linear differential equations

$$(4) \quad \frac{\partial Y}{\partial x} = A(x, t)Y$$

parameterized by  $t$ , where  $A \in \mathrm{gl}_n(\mathcal{O}_{\mathcal{U}}(\{x - \alpha(t)\}))$  depends analytically on  $x$  and  $t$ , as explained in the notation below.

In what follows we will use the words ‘system’ or ‘equation’ indifferently for a matricial equation, that is, a system of equations.

**NOTATION 2.1.**  $\mathcal{U} \subset \mathbb{C}^r$  is an open connected subset containing 0,  $\mathcal{O}_{\mathcal{U}}$  is the ring of analytic functions on  $\mathcal{U}$  of the multi-variable  $t$ , and  $\alpha \in \mathcal{O}_{\mathcal{U}}$ , with  $\alpha(0) = 0$  can be thought of as a moving singularity near 0. Let  $\mathcal{O}_{\mathcal{U}}((x - \alpha(t)))$  denote the ring of formal Laurent series with coefficients in  $\mathcal{O}_{\mathcal{U}}$

$$f(x, t) = \sum_{i \geq m} a_i(t)(x - \alpha(t))^i$$

where  $m \in \mathbb{Z}$  is independent of  $t$ , and let  $\mathcal{O}_{\mathcal{U}}(\{x - \alpha(t)\})$  denote the ring of those  $f(x, t) \in \mathcal{O}_{\mathcal{U}}((x - \alpha(t)))$  that, for each fixed  $t \in \mathcal{U}$ , converge for  $0 < |x - \alpha(t)| < R_t$ , for some  $R_t > 0$ .

Note that in a compact neighbourhood  $\mathcal{N} \subset \mathcal{U}$  of 0, one can choose  $R_t$  to be independent of  $t$ , for  $t \in \mathcal{N}$ .

With these assumptions and notation, we can expand the matrix  $A$  in (4) as

$$A(x, t) = \frac{A_{-m}(t)}{(x - \alpha(t))^m} + \frac{A_{-m+1}(t)}{(x - \alpha(t))^{m-1}} + \dots = \sum_{i \geq -m} (x - \alpha(t))^i A_i(t)$$

where  $A_i(t) \in \mathfrak{gl}_n(\mathcal{O}_{\mathcal{U}})$  for all  $i \geq -m$ , and  $m \in \mathbb{N}$  does not depend on  $t$ .

DEFINITION 2.2. *Two parametric equations*

$$\frac{\partial Y}{\partial x} = AY \quad \text{and} \quad \frac{\partial Y}{\partial x} = BY,$$

with  $A, B \in \mathfrak{gl}_n(\mathcal{O}_{\mathcal{U}}(\{x - \alpha(t)\}))$  are equivalent if for some  $P \in \text{GL}_n(\mathcal{O}_{\mathcal{U}}(\{x - \alpha(t)\}))$

$$B = \frac{\partial P}{\partial x} P^{-1} + PAP^{-1}.$$

DEFINITION 2.3. *With notation as before,*

- (1) Equation (4) has simple singular points near 0 if  $m = 1$  and  $A_{-1} \neq 0$  as an element of  $\mathfrak{gl}_n(\mathcal{O}_{\mathcal{U}})$ ,
- (2) Equation (4) has parameterized regular singular points near 0 (notation  $pr_{s_0}$ ) if it is equivalent to an equation with simple singular points near 0.

EXAMPLE 2.4. Let

$$\begin{aligned} A &= \begin{pmatrix} 0 & -3 \\ 0 & 0 \end{pmatrix} \frac{1}{(x-t)^2} + \begin{pmatrix} t & 0 \\ 0 & t-2 \end{pmatrix} \frac{1}{x-t} \\ B &= \begin{pmatrix} t-1 & 0 \\ 0 & t-1 \end{pmatrix} \frac{1}{x-t} \end{aligned}$$

These equations are equivalent via

$$P = \begin{pmatrix} \frac{1}{x-t} & \frac{-1}{(x-t)^2} \\ 0 & x-t \end{pmatrix}$$

and since the latter has simple singular points near 0, the first equation has parameterized regular singular points near 0.

In analogy to the non-parameterized case, solutions of an equation (4) with parameterized regular singularities near 0 have “uniformly” a moderate growth as  $x$  gets near  $\alpha(t)$  and  $t$  tends to 0 (cf. [32], Cor. 2.6).

PROPOSITION 2.5. *Assume that Equation (4) has regular singular points near 0. Then there is an open connected subset  $\mathcal{U}'$  of  $\mathcal{U}$  such that*

- 1) Equation (4) has a solution  $Y$  of the form

$$(5) \quad Y(x, t) = \left( \sum_{i \geq i_0} (x - \alpha(t))^i Q_i(t) \right) (x - \alpha(t))^{\tilde{A}(t)}$$

with  $\tilde{A} \in \mathfrak{gl}_n(\mathcal{O}_{\mathcal{U}'})$  and  $Q_i \in \mathfrak{gl}_n(\mathcal{O}_{\mathcal{U}'})$  for all  $i \geq i_0$ ,

- 2) for any  $r$ -tuple  $(m_1, \dots, m_r)$  of non-negative integers there is an integer  $N$  such that for any fixed  $t \in \mathcal{U}'$  and any sector  $S_t$  from  $\alpha(t)$  in the complex plane, of opening less than  $2\pi$ ,

$$\lim_{\substack{x \rightarrow \alpha(t) \\ x \in S_t}} (x - \alpha(t))^N \frac{\partial^{m_1 + \dots + m_r} Y(x, t)}{\partial^{m_1} t_1 \dots \partial^{m_r} t_r} = 0.$$

Solutions of parameterized differential equations with *irregular* singularities have been studied by Babbitt and Varadarajan in [2], by Schäfer in [44], and more recently by Dreyfus in [21]. Assuming 0 is a (non-moving) irregular singularity, these authors gave a condition on the exponential part of a formal solution in its usual form

$$\hat{Y}(z) = \hat{H}(z)z^J e^Q$$

ensuring that the coefficients of the formal series  $\hat{H}(z)$  depend analytically on the multi-parameter.

### 3. PPV-theory and monodromy

From the beginning of Picard-Vessiot theory in the nineteenth century, monodromy has been closely related to the ‘group of transformations’ of linear differential equations, now called the Picard-Vessiot group. More information about the history of the monodromy group and the Picard-Vessiot group can be found in [12] and [53].

**3.1. Classical Picard-Vessiot theory and monodromy.** In classical PV-theory it is commonly admitted that the “monodromy matrices belong to the differential Galois group”, which is in particular true for a differential equation (1) over the base-field  $\mathbb{C}(x)$ , but which does not hold over  $\overline{\mathbb{Q}}(x)$  though. Moreover, if (1) has regular singular points only, Schlesinger’s theorem (cf. [45], § 159,160, [40] Th.5.8) tells us that the monodromy matrices generate a Zariski-dense subgroup of the differential Galois group over  $\mathbb{C}(x)$ . For instance, in Example 1.1 above:

$$\frac{dy}{dx} = \frac{t}{x}y$$

let  $t$  denote a constant non-zero complex number. This equation has two regular singular points, at 0 and  $\infty$ . With respect to the solution  $x^t$  (for a given determination of  $\log x$ ) the monodromy ‘matrices’ with respect to 0 and  $\infty$  are the scalars  $m_0 = e^{2\pi it}$  and  $m_\infty = e^{-2\pi it}$ . It is easy to see that the Zariski closure in  $\mathbb{C}^*$  of the subgroup generated by  $m_0$  (or  $m_\infty$ ) is the PV-group over  $\mathbb{C}$  given above ( $\mathbb{C}^*$  or a finite cyclic group).

If  $t \in \overline{\mathbb{Q}}$ , what happens over the differential field  $\overline{\mathbb{Q}}(x)$ ? The monodromy scalars  $e^{\pm 2\pi it}$  may be transcendental in this case and hence not belong to the PV-group, which is a subgroup of  $\overline{\mathbb{Q}}^*$ . But the results given earlier show that the PV-group is defined by the same equation in  $\mathbb{C}^*$  or  $\overline{\mathbb{Q}}^*$  respectively, whether we consider  $t \in \mathbb{C}^*$  or  $t \in \overline{\mathbb{Q}}^*$ . The following example too illustrates the importance of the base field.

EXAMPLE 3.1.

$$\frac{dY}{dx} = \begin{pmatrix} 1/x & 1 \\ 0 & 0 \end{pmatrix} Y.$$

This equation has two regular singular points, one Fuchsian at 0, one at  $\infty$ . With respect to the fundamental solution

$$\begin{pmatrix} x & x \log x \\ 0 & 1 \end{pmatrix}$$

the monodromy matrix at 0 is

$$M = \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}.$$

If we consider the equation over  $\overline{\mathbb{Q}}(x)$ , clearly  $M$  does not belong to the PV-group over  $\overline{\mathbb{Q}}(x)$  since it has a transcendental entry.

To adjust Schlesinger's result to this situation we use the following result (cf. [32], Prop. 3.1 and Cor. 3.2)

**PROPOSITION 3.2.** *Let  $C_0 \subset C_1$  be algebraically closed fields and  $k_0 = C_0(x)$ ,  $k_1 = C_1(x)$  be differential fields where  $c' = 0$  for all  $c \in C_1$  and  $x' = 1$ . Let*

$$(6) \quad Y' = AY$$

*be a differential equation with  $A \in \mathfrak{gl}_n(k_0)$ . If  $G(C_0) \subset \mathrm{GL}_n(C_0)$  is the PV-group over  $k_0$  of Equation (6) with respect to some fundamental solution, where  $G$  is a linear algebraic group defined over  $C_0$ , then  $G(C_1)$  is the PV-group of (6) over  $k_1$ , with respect to some fundamental solution.*

For instance, on Example 3.1, we easily see that the PV-group over  $\overline{\mathbb{Q}}(x)$  is

$$G = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \lambda \in \overline{\mathbb{Q}} \right\}$$

and the PV-group over  $\mathbb{C}(x)$  is the group of  $\mathbb{C}$ -points of  $G$

$$G(\mathbb{C}) = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \lambda \in \mathbb{C} \right\}.$$

The monodromy matrices do belong to the PV-group, after extending scalars.

**COROLLARY 3.3.** *Assume in Equation (6) that  $A \in \mathfrak{gl}_n(C_0(x))$  where  $C_0$  is some algebraically closed subfield of  $\mathbb{C}$ . Assuming 0 is a non-singular point, let us fix it as the base-point of  $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S)$ , where  $S$  is the set of singular points of (6) on  $\mathbb{P}^1(\mathbb{C})$ . Let  $G(C_0)$  be the PV-group of (6) over  $C_0(x)$ , where  $G$  is a linear algebraic group defined over  $C_0$ . If  $C_1$  is any algebraically closed subfield of  $\mathbb{C}$  containing  $C_0$  and the entries of the monodromy matrices, then the monodromy matrices are elements of the PV-group  $G(C_1)$  of (6) over  $C_1(x)$ .*

**3.2. Monodromy matrices in the PPV-group.** In PPV-theory too, the equation may have coefficients in some differentially closed field and the entries of the parameterized monodromy matrices not belong to this field.

In [32] we proved a result similar to Proposition 3.2 for parameterized Picard-Vessiot extensions. Consider equations of the form

$$(7) \quad \partial_x Y = A(x, t)Y$$

where  $A(x, t) \in \mathfrak{gl}_n(\mathcal{O}_{\mathcal{U}}(x))$  and  $t = (t_1, \dots, t_r) \in \mathcal{U}$  for some domain  $\mathcal{U} \subset \mathbb{C}^r$ . Denoting differentiation with respect to  $x, t_1, \dots, t_r$  by  $\partial_x, \partial_{t_1}, \dots, \partial_{t_r}$  respectively, let  $\Delta = \{\partial_x, \partial_{t_1}, \dots, \partial_{t_r}\}$  and  $\Delta_t = \{\partial_{t_1}, \dots, \partial_{t_r}\}$ .

Let  $C$  be a  $\Delta_t$ -differentially closed extension of some field of functions that are analytic on some domain of  $\mathbb{C}^r$  and let  $\partial_{t_i}$  denote for each  $i$  the derivation extending  $\partial_{t_i}$ . We consider the  $\Delta$ -differential field structure on  $k = C(x)$  given by  $\partial_x(x) = 1, \partial_{t_i}(x) = 0$  for each  $i$  and  $\partial_x(c) = 0$  for all  $c \in C$ , and we assume that  $A \in \mathfrak{gl}_n(k)$ .

PROPOSITION 3.4. *Let  $C_0 \subset C_1$  be differentially closed  $\Delta_t$ -fields as  $C$  above, inducing a  $\Delta$ -field structure on  $k_0 = C_0(x)$  and  $k_1 = C_1(x)$ . Let*

$$(8) \quad \partial_x Y = AY$$

*be a differential equation with  $A \in \mathfrak{gl}_n(k_0)$ . If  $G(C_0) \subset \mathrm{GL}_n(C_0)$  is the PPV-group over  $k_0$  of Equation (8) with respect to some fundamental solution, where  $G$  is a linear differential algebraic group defined over the differential  $\Delta_t$ -field  $C_0$ , then  $G(C_1)$  is the PPV-group over  $k_1$  of (8) with respect to some fundamental solution.*

Let us define the parameterized monodromy matrices, which belong to the PPV-group in the same sense as in the non-parameterized case, after extending the base-field.

Let  $\mathcal{D}$  be an open subset of  $\mathbb{P}^1(\mathbb{C})$  with  $0 \in \mathcal{D}$ . Assume that  $\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{D}$  is the union of  $m$  disjoint disks  $D_i$  and that for each  $t \in \mathcal{U}$ , Equation (7) has a unique singular point in  $D_i$ . Let  $\gamma_i$ ,  $i = 1, \dots, m$  be the elementary loops generating  $\pi_1(\mathcal{D}, 0)$ . Let us fix a fundamental solution  $Z_0$  of (7) in the neighborhood of 0 and define, for each fixed  $t \in \mathcal{U}$ , the monodromy matrices of (7) with respect to this solution and the  $\gamma_i$ . These matrices, which depend on  $t$ , are by definition the *parameterized monodromy matrices* of Equation (7).

To prove that the monodromy matrices belong to the PPV-group we need, as in the non-parameterized case, to perform ‘analytic continuation’ of a polynomial expression  $P(Z_0)$  in the entries of  $Z_0$ , where  $P$  is a polynomial with coefficients in  $C_0(x)$ , over some differentially closed field  $C_0$  not contained in  $\mathbb{C}$ . The following result of Seidenberg [46, 47] gives these coefficients, and hence  $P(Z_0)$ , an existence as analytic functions.

THEOREM 3.5 (Seidenberg). *Let  $\mathbb{Q} \subset \mathcal{K} \subset \mathcal{K}_1$  be finitely generated differential extensions of the field of rational numbers  $\mathbb{Q}$ , and assume that  $\mathcal{K}$  consists of meromorphic functions on some domain  $\Omega \in \mathbb{C}^r$ . Then  $\mathcal{K}_1$  is isomorphic to a field  $\mathcal{F}$  of functions that are meromorphic on a domain  $\Omega_1 \subset \Omega$ , such that  $\mathcal{K}|_{\Omega_1} \subset \mathcal{F}$ .*

This leads to the expected analogue of Corollary 3.3:

THEOREM 3.6. *Assume in Equation (7) that  $A \in \mathfrak{gl}_n(C_0(x))$ , where  $C_0$  is any differentially closed  $\Delta_t$ -field containing  $\mathbb{C}$  and let  $C_1$  be any differentially closed  $\Delta_t$ -field containing  $C_0$  and the entries of the parameterized monodromy matrices of Equation (7) with respect to a fundamental solution of (7). Then the parameterized monodromy matrices belong to  $G(C_1)$ , where  $G$  is the PPV-group of (7) over the  $\Delta$ -field  $C_0(x)$ .*

**3.3. A parameterized version of Schlesinger’s theorem.** Consider a family of equations

$$(9) \quad \frac{\partial Y}{\partial x} = A(x, t)Y$$

where the entries of  $A$  are rational in  $x$ , and analytic in  $t$  in some open subset  $\mathcal{U}$  of  $\mathbb{C}^r$ . Let as before  $\mathcal{D}$  be an open subset of  $\mathbb{P}^1(\mathbb{C})$  with  $0 \in \mathcal{D}$ . Assume that  $\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{D}$  is the union of  $m$  disjoint disks  $D_i$  and that for each  $t \in \mathcal{U}$ , Equation (9) has a unique singular point  $\alpha_i(t)$  in each  $D_i$ , and no singular points otherwise. Let  $\gamma_i$ ,  $i = 1, \dots, m$  be the elementary loops generating  $\pi_1(\mathcal{D}, 0)$ . Locally at 0 we can fix a fundamental solution  $Z_0$ , analytic in  $\mathcal{V} \times \mathcal{U}$  where  $\mathcal{V}$  is neighbourhood of 0 in

$\mathcal{D} \subset \mathbb{C}$  and  $\mathcal{U}$  a neighbourhood of 0 in  $\mathbb{C}^r$ . Let as before  $\Delta = \{\partial_x, \partial_{t_1}, \dots, \partial_{t_r}\}$  and  $\Delta_t = \{\partial_{t_1}, \dots, \partial_{t_r}\}$ .

In [32] we proved the following parameterized analogue of Schlesinger's theorem.

**THEOREM 3.7.** *With notation and assumptions as before, assume that Equation (9) has parameterized regular singularities only, near each  $\alpha_i(0)$ ,  $i = 1, \dots, m$ . Let  $k$  be a differentially closed  $\Delta_t$ -field containing the  $x$ -coefficients of the entries of  $A$ , the singularities  $\alpha_i(t)$  of (9) and the entries of the parameterized monodromy matrices with respect to  $Z_0$ . Then the parameterized monodromy matrices generate a Kolchin-dense subgroup of  $G(k)$ , where  $G$  is the PPV-group of (9) over  $k(x)$ .*

**PROOF.** To prove this theorem it is sufficient, by the Galois correspondance of PPV-theory, to show that any element of the PPV-extension  $k(x)\langle Z_0 \rangle$  ( $\Delta$ -differentially generated by a fundamental solution  $Z_0$ ) that is left invariant by the action of the parameterized monodromy matrices, is an element of the base-field  $k(x)$ . Fix such an  $f \in k(x)\langle Z_0 \rangle$ , invariant by all the parameterized monodromy matrices. The idea of the proof is the following. Let  $\mathcal{F}_0$  be the differential  $\Delta_t$ -subfield of  $k$  generated over  $\mathbb{Q}$  by the  $x$ -coefficients of  $A$ , the singular points  $\alpha_i(t)$  and the entries of the parameterized monodromy matrices (with respect to the elementary loops around the  $\alpha_i(t)$ ). Let further  $\mathcal{F}_1$  denote any  $\Delta_t$ -subfield of  $k$  containing  $\mathcal{F}_0$  such that  $f \in \mathcal{F}_1(x)\langle Z_0 \rangle$ . By Seidenberg's theorem 3.5, we can see  $f$  as a meromorphic function on a suitable domain of the  $(x, t)$ -space. Since for each fixed  $t$ , the function  $f$  is invariant by the monodromy matrices and has moreover moderate growth at each singular point by Prop. 2.5, it is indeed a rational function of  $x$ . Note that, as in the non-parameterized case, since  $f$  is single-valued, it has an isolated pole at each singular point of the equation (cf. [27], Preparation Theorem 18.2 p.118). To show that it is globally a rational function of  $x$ , we apply the lemma below, inspired by a result of R. Palais [39].  $\square$

**LEMMA 3.8.** *Let  $\mathcal{F}$  be a  $\Delta$ -field of functions that are meromorphic on  $\mathcal{V} \times \mathcal{U}$  where  $\mathcal{V} \subset \mathbb{C}$  and  $\mathcal{U} \subset \mathbb{C}^r$  are open connected sets, and let  $C_x = \{u \in \mathcal{F} \mid \partial_x u = 0\}$ . Furthermore assume  $x \in \mathcal{F}$ . Let  $f \in \mathcal{F}$  be such that  $f(x, t) \in \mathbb{C}(x)$  for each  $t \in \mathcal{U}$ . Then for some  $m \in \mathbb{N}$ , there exist  $a_0, \dots, a_m, b_0, \dots, b_m \in C_x$  such that*

$$f(x, t) = \frac{\sum_{i=0}^m a_i x^i}{\sum_{i=0}^m b_i x^i}$$

#### 4. PPV-characterization of isomonodromy

Let us first recall that classical differential Galois theory, or PV-theory, extends easily and naturally to differential fields with several derivations. More precisely, let  $k$  be a  $\Delta$ -differential field with derivations  $\Delta = \{\partial_0, \partial_1, \dots, \partial_r\}$ , and consider a linear system of equations

$$(10) \quad \begin{cases} \partial_0 Y &= A_0 Y \\ \partial_1 Y &= A_1 Y \\ &\vdots \\ \partial_r Y &= A_r Y \end{cases}$$

where  $A_0, A_1, \dots, A_r \in \text{gl}(n, k)$ .



Assuming the subfield of  $\Delta$ -constants  $C$  of  $k$  is algebraically closed, for each system (10) there is a unique PV-extension  $K$  of  $k$ , that is, a  $\Delta$ -differential extension of  $k$  generated by the entries of a fundamental solution of (10) with no new  $\Delta$ -constants. The corresponding PV-group of differential  $k$ -automorphisms of  $K$  is a linear algebraic group  $G \subset \mathrm{GL}(n, C)$ , unique up to differential isomorphism, and satisfying Galois correspondence.

**4.1. Integrable systems.** The notion of integrability has a nice interpretation in terms of PPV-theory. Integrability, over abstract differential fields, has the same definition as over fields of analytic functions (cf. [17]).

DEFINITION 4.1. *With notation as above*

- (1) *the differential system (10) is integrable if*

$$\partial_i A_j - \partial_j A_i = [A_i, A_j]$$

*for all  $0 \leq i, j \leq r$ , where  $[ , ]$  denotes the Lie bracket,*

- (2) *an equation*

$$\partial_0 Y = AY, \quad A \in \mathrm{gl}(n, k)$$

*is completely integrable if it can be completed into a system (10) with  $A_0 = A$ .*

For completely integrable equations, PV-theory and PPV-theory get close (cf. [17]), Lemma 9.9).

LEMMA 4.2. *With notation as above, assume the field  $k_0$  of  $\partial_0$ -constants of  $k$  is  $\Delta$ -differentially closed, and let*

$$(11) \quad \partial_0 Y = AY, \quad A \in \mathrm{gl}(n, k)$$

*be a completely integrable system, completable into an integrable system (10) as above. Then any PV-extension of  $k$  for (10) is a PPV-extension of  $k$  for (11).*

The proof of this lemma relies on the fact that a differentially closed field is *a fortiori* algebraically closed, and that the field of constants of an algebraically closed differentially field is itself algebraically closed. This lemma was used by Cassidy and Singer to give the following PPV-characterization of integrability (cf. [17], Prop. 3.9).

PROPOSITION 4.3 (Cassidy-Singer). *With notation as above, assume  $k_0$  is differentially closed, and let  $C \subset k_0$  denote the subfield of  $\Delta$ -constants of  $k$ .*

- (1) *Equation (11) is completely integrable if and only if its PPV-group over  $k$  is conjugate in  $\mathrm{GL}(n, k_0)$  to the group  $G(C)$  of  $C$ -points of some linear algebraic group defined over  $C$ .*
- (2) *In particular, (1) holds if  $A \in \mathrm{gl}(C)$ .*

**4.2. Isomonodromy.** Let us again consider the case of differential fields containing analytic functions. We consider as in Section 3.3 a parameterized system

$$(12) \quad \partial_x Y = A(x, t)Y$$

where the entries of  $A$  are analytic on  $\mathcal{D} \times \mathcal{U}$  for some open subset  $\mathcal{U} \subset \mathbb{C}^r$  containing 0 and some open subset  $\mathcal{D}$  of  $\mathbb{P}^1(\mathbb{C})$  containing 0 and such that  $\pi_1(\mathcal{D}, 0)$  is generated by elementary loops  $\gamma_1, \dots, \gamma_m$ . More precisely we assume that  $\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{D}$  is the union of  $m$  disjoint disks  $D_i$  and that for each  $t \in \mathcal{U}$ , Equation (12) has a unique singular point  $\alpha_i(t)$  in each  $D_i$ , and no singular points otherwise.

DEFINITION 4.4. Equation (12) is isomonodromic on  $\mathcal{D} \times \mathcal{U}$  if there are constant matrices  $M_1, \dots, M_m \in \mathrm{GL}(n, \mathbb{C})$  such that for each fixed  $t \in \mathcal{U}$  there is a local fundamental solution  $Y_t$  of (12) at 0 such that analytic continuation  $Y_t^{\gamma_i}$  of  $Y_t$  along  $\gamma_i$  yields

$$Y_t^{\gamma_i} = Y_t M_i$$

for  $i = 1, \dots, m$ .

Note that  $Y_t$  may *a priori* not be analytic in  $t$ . Nevertheless, following a proof by Andrey Bolibrukh in the Fuchsian case (cf. [8]), one can show the existence of such a solution  $Y_t$  which is analytic in  $t$ , using in particular the fact that  $\mathcal{U}$  is a Stein variety, on which any topological trivial (analytic) bundle is analytically trivial (cf. [15]).

A useful criterion for isomonodromy is the following.

THEOREM 4.5 (Sibuya [50]). Consider an equation (12) with notation and assumptions as above.

- (1) Equation (12) is isomonodromic on  $\mathcal{D} \times \mathcal{U}$  if and only if it is completely integrable, that is, part of an integrable system

$$\begin{cases} \partial_0 Y = A_0 Y \\ \partial_1 Y = A_1 Y \\ \vdots \\ \partial_r Y = A_r Y \end{cases}$$

with  $A_0 = A$  and analytic  $A_i$  on  $\mathcal{D} \times \mathcal{U}$  for all  $i$ .

- (2) Assume (12) is isomonodromic. If moreover  $A$  is rational in  $x$  and Equation (12) has parameterized regular singular points only, then the entries of all  $A_i$  are rational in  $x$ .

In [17] Cassidy and Singer give an algebraic criterion for isomonodromy using PPV-theory. Let as before  $\Delta = \{\partial_x, \partial_{t_1}, \dots, \partial_{t_r}\}$  and  $\Delta_t = \{\partial_{t_1}, \dots, \partial_{t_r}\}$  denote the partial differentiation with respect to  $x$  and the multi-parameter  $t$ .

THEOREM 4.6 (Cassidy-Singer). Consider an equation

$$\partial_x Y = A(x, t)Y$$

as before, where  $A$  has entries analytic in  $\mathcal{D} \times \mathcal{U}$ , rational in  $x$ , with parameterized regular singularities only, one in each disk  $D_i$ . Let  $k = C_0(x)$ , where  $C_0$  is a  $\Delta_t$ -differential closure of the field generated over  $\mathbb{C}(t_1, \dots, t_r)$  by the  $x$ -coefficients of the entries (which are rational functions of  $x$ ) of  $A$ . This equation is isomonodromic if and only if its PPV-group over  $k$  is conjugate in  $\mathrm{GL}(n, C_0)$  to a linear algebraic subgroup of  $\mathrm{GL}(n, \mathbb{C})$ .

The proof of this theorem relies on Sibuya's criterion and Proposition 4.3.

## 5. Projective isomonodromy

Consider as before a parameterized equation

$$(13) \quad \partial_x Y = A(x, t)Y$$

on  $\mathcal{D} \times \mathcal{U}$  with  $m$  isolated singular points, each in a disk  $D_i$  such that  $\mathcal{D} = \mathbb{P}^1(\mathbb{C}) \setminus \bigcup_{i=1}^m D_i$ . We are now considering a special case of so-called *monodromy evolving*

deformations, which has been studied on the classical example of the Darboux-Halphen equation by Chakravarty and Ablowits [18] and Ohya ( [37], [38]).

DEFINITION 5.1. *Equation (13) is projectively isomonodromic if there are constant matrices  $\Gamma_1, \dots, \Gamma_m \in \mathrm{GL}(n, \mathbb{C})$  and analytic functions  $c_1, \dots, c_m \in \mathcal{O}_{\mathcal{U}}$  such that for each fixed  $t \in \mathcal{U}$  there is locally at 0 a fundamental solution  $Y_t$  of (13) such that for each  $i$  the parameterized monodromy matrix of (13) with respect to  $Y_t$  and the loop  $\gamma_i$  is*

$$c_i(t)\Gamma_i.$$

As in the isomonodromic case, the solution  $Y_t$  may not be analytic in  $t$  and in [33] we mimic Bolibrukh's proof to show the existence of such a particular solution that is analytic in  $t$ . We need such a solution to interpret projective isomonodromy algebraically in terms of PPV-theory.

In the special case of a Fuchsian parameterized equation

$$(14) \quad \partial_x Y = \sum_{i=1}^m \frac{A_i(t)}{x - \alpha_i(t)} Y$$

projective isomonodromy is related to isomonodromy in a natural way (cf. [33]).

PROPOSITION 5.2. *Equation (14) is projectively isomonodromic if and only if for each  $i$*

$$A_i(t) = B_i(t) + b_i(t)I$$

where  $b_i$  and the entries of  $B_i$  are analytic on  $\mathcal{U}$  and such that the equation

$$\partial_x Y = \sum_{i=1}^m \frac{B_i(t)}{x - \alpha_i(t)} Y$$

is isomonodromic.

For general equations (13) with parameterized regular singularities we give in [33] an algebraic characterization of projective isomonodromy in terms of their PPV-group.

THEOREM 5.3. *With notation as before, consider a parameterized equation*

$$(15) \quad \partial_x Y = A(x, t)Y$$

where  $A$  has entries analytic in  $\mathcal{D} \times \mathcal{U}$ , rational in  $x$ , and assume that this equation has parameterized regular singularities only, one in each disk  $D_i$ . Let  $k = k_0(x)$ , where  $k_0$  is a  $\Delta_t$ -differential closure of the field generated over  $\mathbb{C}(t_1, \dots, t_r)$  by the  $x$ -coefficients of the rational functions entries of  $A$ . Then this equation is projectively isomonodromic if and only if its PPV-group over  $k$  is conjugate in  $\mathrm{GL}(n, k_0)$  to a subgroup of

$$\mathrm{GL}(n, \mathbb{C}) \cdot k_0 I \subset \mathrm{GL}(n, k_0)$$

where  $k_0 I$  is the subgroup of scalar matrices of  $\mathrm{GL}(n, k_0)$ .

Combining topological arguments in both the Kolchin and the Zariski topology, and using Schur's lemma we get a corollary of this result for *absolutely irreducible* equations over  $k$ , that is, equations that are irreducible over any finite extension of  $k$ . We recall that an equation is said to be irreducible if the corresponding differential polynomial is irreducible (it has no factor of strictly less

order), equivalently if its differential Galois group acts irreducibly on its solution space in any Picard-Vessiot extension.

**COROLLARY 5.4.** *Let  $A, k_0$  and  $k$  be as in Theorem 5.3. If Equation (15) is absolutely irreducible, then it is projectively isomonodromic if and only if the commutator subgroup  $(G, G)$  of its PPV-group  $G$  is conjugate in  $\mathrm{GL}(n, k_0)$  to a subgroup of  $\mathrm{GL}(n, \mathbb{C})$ .*

## 6. The Darboux-Halphen equation

The results of the previous section are well illustrated on the Darboux-Halphen equation. This equation describes projective isomonodromy in the same way as the Schlesinger equation accounts for isomonodromy (of the Schlesinger type) for parameterized Fuchsian systems. The Darboux-Halphen V equation

$$(DH\ V) \quad \begin{cases} \omega'_1 = & \omega_2\omega_3 & - & \omega_1(\omega_2 + \omega_3) & + & \phi^2 \\ \omega'_2 = & \omega_3\omega_1 & - & \omega_2(\omega_3 + \omega_1) & + & \theta^2 \\ \omega'_3 = & \omega_1\omega_2 & - & \omega_3(\omega_1 + \omega_2) & - & \theta\phi \\ \phi' = & \omega_1(\theta - \phi) & - & \omega_3(\theta + \phi) \\ \theta' = & -\omega_2(\theta - \phi) & - & \omega_3(\theta + \phi), \end{cases}$$

occurs in physics as a reduction of the selfdual Yang-Mills equation (SDYM). For  $\theta = \phi$ , (DH V) is equivalent to Einstein's selfdual vacuum equations. For  $\theta = \phi = 0$ , it is Halphen's original equation (H II), solving a geometry problem of Darboux about orthogonal surfaces.

Contrary to other SDYM reductions such as the Painlevé equations, (DH V) does not satisfy the Painlevé property, since it has a boundary of movable essential singularities. It is therefore not likely to rule isomonodromy.

**6.1. History of the DH-equation.** Halphen's equation (H II) goes back to Darboux's work ([19], [20]) on orthogonal systems of surfaces. Darboux's original problem was the following.

*Problem 1:* What condition on a given pair  $(\mathcal{F}_1, \mathcal{F}_2)$  of orthogonal families of surfaces in  $\mathbb{R}^3$  implies that there exists a family  $\mathcal{F}_3$  such that  $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$  is a triorthogonal system of pairwise orthogonal families?

In [19] Darboux gives a necessary and sufficient condition on  $(\mathcal{F}_1, \mathcal{F}_2)$  to solve the problem: that the intersection of any surfaces  $S_1 \in \mathcal{F}_1$  and  $S_2 \in \mathcal{F}_2$  be a curvature line of both  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . The necessary condition was already known as Dupin's theorem (1813).

*Problem 2:* What condition on its parameter  $u = \varphi(x, y, z)$  implies that a one-parameter family  $\mathcal{F}$  of surfaces in  $\mathbb{R}^3$  belongs to a *triorthogonal* system  $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ , of three pairwise orthogonal families?

In [20] Darboux found and solved an order three partial differential equation satisfied by  $u$  and obtained, based on previous work by Bonnet and Cayley, the general solution from a particular family of ruled helicoidal surfaces. Élie Cartan [14] later used his exterior differential calculus to prove that Problem 2 has a solution. He also generalized the problem, replacing orthogonality by any prescribed angle, or considering  $p$  pairwise orthogonal families of hypersurfaces in  $p$ -space.

Darboux stated yet another problem on orthogonal surfaces.

*Problem 3:* given two families  $\mathcal{F}_1$  and  $\mathcal{F}_2$  consisting each of parallel surfaces does there exist a family  $\mathcal{F}$  orthogonal to both  $\mathcal{F}_1$  and  $\mathcal{F}_2$  ?

It is an easy exercise to prove that a solution should either consist of planes, or of ruled quadrics. If  $\mathcal{F}$  consists of quadrics *with a center*, these have simultaneously reduced equations:

$$\frac{x^2}{a(u)} + \frac{y^2}{b(u)} + \frac{z^2}{c(u)} = 1$$

which depend on the parameter  $u = \varphi(x, y, z)$  of  $\mathcal{F}$ . One can show that  $\mathcal{F}$  solves Problem 3 if and only if  $a, b, c$  satisfy the *Darboux equation*

$$a(b' + c') = b(c' + a') = c(a' + b')$$

where  $a', b', c'$  are the derivatives with respect to  $u$ . Darboux could not solve the problem though:

*'These equations do not seem to be integrable by known procedures'* (Darboux, 1878).

He gave up on this part of the problem and restricted his study to centerless quadrics. He solved the particular problem with a family  $\mathcal{F}$  of paraboloids

$$\frac{y^2}{\alpha + u} + \frac{z^2}{\alpha - u} = 2x + \alpha \log u$$

and claimed that some surfaces of revolution solved the problem as well.

In 1881 Halphen ([22], [23]) completely solved Darboux's second problem in the following form:

$$(H I) \quad \begin{cases} \omega'_1 + \omega'_2 = \omega_1 \omega_2 \\ \omega'_2 + \omega'_3 = \omega_2 \omega_3 \\ \omega'_3 + \omega'_1 = \omega_3 \omega_1 \end{cases}$$

known as the *Halphen I equation*, and actually solved the more general QHDS (quadratic homogeneous differential system)

$$(H II) \quad \begin{cases} \omega'_1 = a_1 \omega_1^2 + (\lambda - a_1)(\omega_1 \omega_2 + \omega_3 \omega_1 - \omega_2 \omega_3) \\ \omega'_2 = a_2 \omega_2^2 + (\lambda - a_2)(\omega_2 \omega_3 + \omega_1 \omega_2 - \omega_3 \omega_1) \\ \omega'_3 = a_3 \omega_3^2 + (\lambda - a_3)(\omega_3 \omega_1 + \omega_2 \omega_3 - \omega_1 \omega_2) \end{cases}$$

known as the *Halphen II equation*, by means of hypergeometric functions. He considered even more general QHDSs

$$\{\omega'_r = \psi_r(\omega_1, \dots, \omega_l)\}_{r=1, \dots, l}$$

where the  $\psi_r$  are quadratic forms, with some extra symmetry condition. A special example of such QHDS is Equation (DH V) above, and its particular form (H II) which we consider now.

**6.2. Application of PPV-theory to the Darboux-Halphen.** As shown in [37], Equation (H II) is equivalent to a system

$$x'_i = Q_i(x_1, x_2, x_3), \quad i = 1, 2, 3,$$

where  $Q_i(x_1, x_2, x_3) = x_i^2 + a(x_1 - x_2)^2 + b(x_2 - x_3)^2 + c(x_3 - x_1)^2$  for some constants  $a, b, c$ .

Equation (H II) is in fact the integrability condition of the Lax pair

$$(16) \quad \frac{\partial Y}{\partial x} = \left( \frac{\mu I}{(x-x_1)(x-x_2)(x-x_3)} + \sum_{i=1}^3 \frac{\lambda_i C}{x-x_i} \right) Y$$

$$(17) \quad \frac{\partial Y}{\partial t} = \left( \nu I + \sum_{i=1}^3 \lambda_i x_i C \right) Y - Q(x) \frac{\partial Y}{\partial x}$$

where

$$Q(x) = x^2 + a(x_1 - x_2)^2 + b(x_2 - x_3)^2 + c(x_3 - x_1)^2$$

and where  $x_i = x_i(t)$  are parameterized (simple) singularities,  $C$  is a constant traceless  $2 \times 2$  matrix,  $I$  is the identity matrix,  $\mu \neq 0$  and  $\lambda_i$  are constants such that  $\lambda_1 + \lambda_2 + \lambda_3 = 0$  (there is hence no singular point at  $\infty$ ), and the function  $\nu$  is a solution of

$$\frac{\partial \nu}{\partial x} = -\mu \frac{x + x_1 + x_2 + x_3}{(x-x_1)(x-x_2)(x-x_3)}.$$

Note that since the solutions of the latter equation are not rational in  $x$ , Equation (16) is not isomonodromic, by Sibuya's criterion. To describe the monodromy of this equation, let us fix a fundamental solution  $Y$  of the Lax pair at some  $x_0$  not belonging to fixed disks  $D_i$  with centers  $x_i(t)$ , for all  $i$ . Note that  $Y$  must be analytic in both  $x$  and  $t$ . A computation shows that the parameterized monodromy matrix of Equation (16) with respect to  $Y$  and  $x_i(t)$  is

$$M_i(t) = e^{-2\pi\sqrt{-1}\mu \int_{t_0}^t \beta_i(t) dt} e^{2\pi\sqrt{-1}L_i(t_0)}$$

where  $L_i(t)$  is an analytic function of  $t$  such that, for some fundamental solution  $Y_0$  of Equation (16) in the neighbourhood of given non-singular point  $x_0$ , the analytic extension of  $Y_0$  to a neighbourhood of  $x_i(t)$  is

$$Y(t, x) = Y_i(t, x - x_i(t)) \cdot (x - x_i(t))^{L_i(t)}$$

where  $Y_i$  is single-valued. The coefficients  $\beta_i(t)$  are given by

$$\frac{x + \sum_{i=1}^3 x_i}{\prod_{i=1}^3 (x - x_i(t))} = \sum_{i=1}^3 \frac{\beta_i(t)}{x - x_i(t)}.$$

The monodromy matrix is for each  $i$  of the form

$$M_i(t) = c_i(t) M_i(t_0)$$

with

$$c_i(t) = e^{-2\pi\sqrt{-1}\mu \int_{t_0}^t \beta_i(t) dt}, \quad M_i(t_0) = e^{2\pi\sqrt{-1}L_i(t_0)},$$

that is, Equation (16) is projectively isomonodromic. Moreover it is an example of a Fuchsian projectively isomonodromic equation to which Proposition 5.2 applies, since we can write this equation

$$\frac{\partial Y}{\partial x} = \left( \sum_{i=1}^3 \frac{A_i(t)}{(x-x_i)} \right) Y$$

where

$$A_i(t) = B_i(t) + b_i(t)I$$

$$B_i(t) = \lambda_i C, \quad b_i(t) = \frac{\mu}{\prod_{j \neq i} (x_i - x_j)}$$

and where

$$\frac{\partial Y}{\partial x} = \left( \sum_{i=1}^3 \frac{\lambda_i C}{(x - x_i)} \right) Y$$

is clearly isomonodromic.

## 7. Inverse problems

**7.1. A parameterized version of the weak Riemann-Hilbert problem.** In [32] we adapted Bolibrukh's techniques and construction of holomorphic bundles (cf. [1], [4], [5], [6], [7], [11]) to give a parameterized version of the weak Riemann-Hilbert problem.

**THEOREM 7.1.** *Let  $S = \{a_1, \dots, a_s\}$  be a finite subset of  $\mathbb{P}^1(\mathbb{C})$  and  $D$  an open polydisk in  $\mathbb{C}^r$ . Let  $\gamma_1, \dots, \gamma_s$  be generators of  $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S; a_0)$  for some fixed base-point  $a_0 \in \mathbb{P}^1(\mathbb{C}) \setminus S$ , and let  $M_i : D \rightarrow \mathrm{GL}_n(\mathbb{C})$ ,  $i = 1, \dots, s$ , be analytic maps with  $M_1 \cdot \dots \cdot M_s = I_n$ . There exists a parameterized linear differential system*

$$\partial_x Y = A(x, t) Y$$

with  $A \in \mathfrak{gl}_n(\mathcal{O}_{D'}(x))$  for some open polydisk  $D' \subset D$ , with only regular singular points, all in  $S$ , such that for some parameterized fundamental solution, the parameterized monodromy matrix along each  $\gamma_i$  is  $M_i$ . Furthermore, given any  $a_i \in \{a_1, \dots, a_s\}$ , the entries of  $A$  may be chosen to have at worst simple poles at all  $a_j \neq a_i$ .

The proof, as in the non-parameterized case, here relies on a parameterized version of the Birkhoff-Grothendieck theorem (cf. [29], Proposition 4.1; [9], Theorem 2; [10], Theorem A.1).

**7.2. The inverse problem of PPV-theory.** In analogy again with the non-parameterized case, we deduce in [32] the following consequence of the parameterized versions Theorem 3.7 of Schlesinger's theorem and Theorem 7.1 above of the weak Riemann-Hilbert problem. As before, let  $t = (t_1, \dots, t_r)$  be a multi-parameter and  $\Delta_t = \{\partial_{t_1}, \dots, \partial_{t_r}\}$  the corresponding partial derivations. We consider the differential field  $k = k_0(x)$ , where  $k_0$  is a  $\Delta_t$ -differentially closed field containing  $\mathbb{C}(t_1, \dots, t_r)$ , and  $k$  is endowed with the derivations  $\Delta = \{\partial_x, \partial_{t_1}, \dots, \partial_{t_r}\}$ .

**THEOREM 7.2.** *Let  $G$  be a  $\Delta_t$ -linear differential algebraic group defined over  $k_0$  and assume that  $G(k_0)$  contains a finitely generated Kolchin-dense subgroup  $H$ . Then  $G(k_0)$  is the PPV-group of a PPV-extension of  $k = k_0(x)$ .*

The condition in Theorem 7.2, that  $G(k_0)$  contains a finitely generated Kolchin-dense subgroup  $H$ , characterizes indeed those linear differential algebraic groups over  $k_0$  which are PPV-groups. The fact that the condition is also necessary was proved by Dreyfus [21] as a consequence of his parameterized version of Ramis's density theorem (see for example [40] p. 238). Ramis's theorem says that the (local) differential Galois group over  $\mathbb{C}(\{x\})$  (local at 0) of a linear differential system of order  $n$  is the Zariski-closure in  $\mathrm{GL}(n, \mathbb{C})$  of a subgroup finitely generated by the so-called formal monodromy, Stokes matrices and exponential torus, together

also called *generalized monodromy data*, which generalize to irregular singularities the notion of monodromy matrices for regular singularities. Moreover, it can be proved that the (global) differential Galois group over  $\mathbb{C}(x)$  of a linear differential system is the Zariski-closure of the subgroup generated by the finitely many “local” differential Galois groups just mentioned, which can be simultaneously embedded as subgroups in the global PV-group. Dreyfus [21] defines a parameterized version of the generalized monodromy data and gives a parameterized version of this theorem, which in turn gives the converse result of Theorem 7.2 above.

In the non-parameterized case, the solution by Tretkoff and Tretkoff [51] of the differential Galois inverse problem over  $\mathbb{C}(x)$  uses the fact, proved by the same authors, that over an algebraically closed field of characteristic zero, any linear algebraic group is the Zariski closure of some finitely generated subgroup. The latter does not hold though for linear *differential* algebraic groups. This can in particular be seen on the additive group  $\mathbb{G}_a(k_0)$  (using notation as above for the differential field  $k_0$ ) which has the striking property that the Kolchin-closure of any of its finitely generated subgroups is a proper subgroup of  $\mathbb{G}_a(k_0)$  (cf. [32]). In [28] and [17] it is furthermore shown that neither  $\mathbb{G}_a(k_0)$  nor  $\mathbb{G}_m(k_0)$  is the PPV-group of any PPV-extension of  $k_0(x)$ . In [49], Singer proves the following result, using Corollary 7.2.

**THEOREM 7.3.** *With notation as above, a linear algebraic group  $G$  defined over  $k_0$  is a PPV-group of a PPV-extension of  $k_0(x)$  if and only if the identity component of  $G$  has no quotient isomorphic to  $\mathbb{G}_a(k_0)$  or  $\mathbb{G}_m(k_0)$ .*

More recently, Minchenko, Ovchinnikov and Singer [34] gave a characterization of linear unipotent differential algebraic groups that can be realized as PPV-groups.

**THEOREM 7.4** (Minchenko, Ovchinnikov, Singer). *A unipotent linear differential algebraic group  $G$  over  $k_0$  is the Kolchin-closure of a finitely generated subgroup if and only if it has differential type 0.*

The meaning here of “differential type 0” is that a so-called ‘differential dimension’ be finite. The latter is defined as the transcendence degree over  $k_0$  of the ‘differential function field’  $k_0\langle G^0 \rangle$  over  $k_0$  of the identity component  $G^0$  of  $G$ . If  $G \subset \mathrm{GL}(n, k_0)$ , the differential function field of  $G^0$ , denoted  $k_0\langle G^0 \rangle$ , is the quotient-field of  $R/\mathcal{I}$ , where  $R/\mathcal{I}$  is the differential coordinate ring of the group. More precisely,  $R/\mathcal{I}$  is the quotient of the ring of differential polynomials  $k_0\{y_{1,1}, \dots, y_{n,n}\}$  in  $n^2$  differential indeterminates (differential with respect to  $\Delta_i$ ) by the differential ideal  $\mathcal{I}$  of those differential polynomials vanishing on  $G^0$ .

The same authors have also given a characterization in [35] of those reductive linear differential algebraic groups that can occur as PPV-groups over  $k_0(x)$ . In both [34] and [35] the authors give algorithms to determine if the PPV-groups is of the relevant type and give algorithms to compute this group if it is.

## 8. Appendix

Let  $(K, \partial)$  be an ordinary differential field and  $K\{X\}$  the differential ring of differential polynomials in one differential variable. By definition  $K\{X\}$  is the ring  $K[X_0, X_1, \dots, X_n, \dots]$  of polynomials in the indeterminates  $X_0, X_1, \dots, X_n, \dots$ , with the derivation  $\partial$  extended by  $\partial X_i = X_{i+1}$  for all  $i \geq 0$ . In  $K\{X\}$  one writes  $X$  for



$X_0, X'$  for  $X_1$ , and  $X^{(i)} := \partial^{(i)}X$  for all  $X_i$ . The *order*  $o(f)$  of an element  $f \in K\{X\}$  is defined as the least integer  $n$  such that  $f \in K[X_0, X_1, \dots, X_n]$  if  $f \notin K$ , and  $o(f) = -1$  if  $f \in K$ . For basic facts and model theoretic properties of the theory DCF of differential closed fields, we refer for instance to [30], [31], [42].

The following definition is close to the definition of algebraic closedness. It is due to Blum[3], who simplified an earlier definition introduced by Robinson [41].

**DEFINITION 8.1 (Blum).** *The differential field  $(K, \partial)$  is said to be differentially closed if for any  $f, g \in K\{X\}$ ,  $f \notin K$  with  $o(g) < o(f)$ , there is an  $a \in K$  such that  $f(a) = 0$  and  $g(a) \neq 0$ .*

This definition is for instance well illustrated on Example 1.1 above

$$\frac{dy}{dx} = \frac{t}{x}y.$$

Let us show that over  $K(x)$ , where  $K$  is a differentially closed field containing  $\mathbb{C}(t)$ , the obstruction to Galois correspondence vanishes. We recall that the PPV-extension of this equation over  $K(x)$  is  $K(x, x^t, \log x)$  and that an element  $\sigma$  of the PPV-group is defined by

$$\sigma(x^t) = a_\sigma x^t, \quad \sigma(\log x) = \log x + c_\sigma$$

where  $a_\sigma \in K^*$  satisfies

$$a''_\sigma a_\sigma - a'^2_\sigma = 0$$

and

$$c_\sigma = \frac{a'_\sigma}{a_\sigma},$$

and where  $a'_\sigma, a''_\sigma$  are derivatives with respect to the derivation extending  $d/dt$ .

To avoid that  $\log x$  be invariant by the PPV-group (in which case the invariant field of the PPV-group would not be the base-field  $K(x)$ ) we need at least one  $\sigma$  to be such that  $\sigma(\log x) \neq \log x$ , that is, given by  $a_\sigma \in K^*$  such that

$$a''_\sigma a_\sigma - a'^2_\sigma = 0, \quad \frac{a'_\sigma}{a_\sigma} \neq 0.$$

Since  $K$  is differentially closed, such an element exists by Definition 8.1 applied to  $f(X) = X''X - X'^2$  and  $g(X) = X'$ .

The definition of general (non-ordinary) differentially closed fields is due to Kolchin cf. [24] who called them “constrainedly closed”. For ordinary differential fields, the definition below is equivalent to Definition 8.1 above.

**DEFINITION 8.2 (Kolchin).** *Let  $K$  be a  $\Delta$ -differential field, endowed with a finite set  $\Delta$  of commuting derivations on  $K$ . The field  $K$  is  $\Delta$ -differentially closed if it has no proper constrained extensions.*

The definition of constrained extensions is the following.

**DEFINITION 8.3.** *Let  $K$  be a  $\Delta$ -differential field. A differential extension  $L$  of  $K$  is said to be constrained if for any finite family of elements  $(\eta_1, \dots, \eta_s)$  of  $L$  there is a  $\Delta$ -differential polynomial  $P \in K\{y_1, \dots, y_s\}$  such that  $P(\eta_1, \dots, \eta_s) \neq 0$  whereas  $P(\zeta_1, \dots, \zeta_s) = 0$  for any non-generic differential specialization  $(\zeta_1, \dots, \zeta_s)$  of  $(\eta_1, \dots, \eta_s)$  over  $K$ .*

In Kolchin's terminology, a differential specialization  $\zeta = (\zeta_1, \dots, \zeta_s)$  of  $\eta = (\eta_1, \dots, \eta_s)$  in some extension of  $K$  is *generic* if the defining ideals of  $\zeta$  and  $\eta$  in  $K\{y_1, \dots, y_s\}$  are the same. We refer to Kolchin's original work for details about these notions (cf. [24], [25], [26]). The differential closure is defined in a similar way as the algebraic closure.

**DEFINITION 8.4.** *Let  $K$  be a  $\Delta$ -differential field. A differential closure of  $K$  is a differential, differentially closed extension of  $K$  which can be embedded in any given differential, differentially closed extension of  $K$ .*

**THEOREM 8.5.** *A differential field  $K$  has a unique differential closure.*

This result was proved by Morley [36], Blum [3], Shelah [48] and Kolchin [24]. Unlike the algebraic closure though, the differential closure fails to be minimal, even in characteristic 0. Although it had been conjectured by some authors to be minimal (cf. [43]), Kolchin, Rosenlicht, and Shelah independently proved that it is not. Shelah [48] in particular proved that the ordinary differential closure  $\tilde{\mathbb{Q}}$  of  $\mathbb{Q}$  is not minimal by exhibiting an infinite, strictly decreasing sequence of differentially closed intermediate differential extensions of  $\mathbb{Q}$  in  $\tilde{\mathbb{Q}}$ .

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## References

1. D. V. Anosov and A. A. Bolibruch, The Riemann-Hilbert Problem, Vieweg, Braunschweig, Wiesbaden, 1994.
2. D. G. Babbitt and V. S. Varadarajan, Deformations of nilpotent matrices over rings and reduction of analytic families of differential equations, *Memoirs AMS* 55 (325), 1985.
3. L. Blum, Generalized algebraic structures: A model theoretic approach. Ph.D. dissertation, Massachusetts Institute of Technology, Cambridge, MA, 1968.
4. A. A. Bolibruch, The Riemann-Hilbert problem, *Russian Math. Surveys*, 45, 1-47, 1990.
5. A. A. Bolibruch, On sufficient conditions for the positive solvability of the Riemann-Hilbert problem, *Math. Notes. Acad. Sci. USSR*, 51 (1), 110-117, 1992.
6. A. A. Bolibruch, On an analytic transformation to the standard Birkhoff form, *Proc. Steklov Inst. Math.* 203 (3), 29-35, 1995.
7. A. A. Bolibruch, The 21st Hilbert Problem for Linear Fuchsian Systems, *Proc. Steklov Inst. Math.* 206 (5), 1-145, 1995.
8. A. A. Bolibruch, On Isomonodromic Deformations of Fuchsian Systems *J. Dynam. Contr. Sys.*, 3 (4), 589-604, 1997.
9. A. A. Bolibruch, Inverse problems for linear differential equations with meromorphic coefficients, Isomonodromic deformations and applications in physics (Montréal, QC, 2000), CRM Proc. Lecture Notes 31, 3-25, 2002.
10. A. A. Bolibruch and A. R. Its and A. A. Kapaev, On the Riemann-Hilbert-Birkhoff inverse monodromy problem and the Painlevé equations. *Algebra i Analiz*, 16(1), 121-162, 2004.
11. A. A. Bolibruch and S. Malek and C. Mitschi, On the generalized Riemann-Hilbert problem with irregular singularities, *Expositiones Mathematicae*, 24(3), 235-272, 2006.
12. A. Borel, *Essays in the History of Lie Groups and Algebraic Groups* American mathematical Society, 2001.
13. A. Buium *Differential Algebraic Groups of Finite Dimension* Springer Lecture Notes in Math. 1506, 1992.
14. É. Cartan, *Les systèmes différentiels extérieurs et leurs applications géométriques*, Exposés de géométrie XII, Hermann, Paris, 1945.
15. H. Cartan, *Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes*, Hermann, Paris, 1961.

16. P. J. Cassidy, Differential algebraic groups *American Journal of Mathematics*, 94:891-954, 1972.
17. P. J. Cassidy, M. F. Singer, Galois Theory of parameterized Differential Equations and Linear Differential Algebraic Groups, *Differential Equations and Quantum Groups*, D. Bertrand et. al., eds., IRMA Lectures in Mathematics and Theoretical Physics 9, 113-157, 2006.
18. S. Chakravarty, M. J. Ablowitz, Integrability, monodromy evolving deformations, and self-dual Bianchi IX systems, *Physical Review Letters* 76(6), 857-860, 1996.
19. G. Darboux, Systèmes orthogonaux, *Ann. Sc. É.N.S.*, 1e série, tome 3, 97 -141, 1866.
20. G. Darboux, Mémoire sur la théorie des coordonnées curvilignes, et des systèmes orthogonaux, *Ann. Sc. É.N.S.*, 2e série, tome 7, 101-150, 227-260, 275-348, 1878.
21. T. Dreyfus, A parameterized density theorem in differential Galois theory *Pacific J. Math.* 271(1) 87-141, 2014.
22. G. H. Halphen, Sur un système d'équations différentielles, *C. R. Acad. Sci.* 92, 1101-1103, 1881.
23. G. H. Halphen, Sur certains systèmes d'équations différentielles, *C. R. Acad. Sci.* 92, 1404-1406, 1881.
24. E. Kolchin, Constrained extensions of differential fields, *Advances in Math.* 12 (2), 141-170, 1974.
25. E. R. Kolchin, *Differential Algebra and Algebraic Groups*, Academic Press, 1976.
26. E. R. Kolchin, *Differential algebraic groups* Academic Press, New York, 1985
27. M. Kuga, Galois'Dream : Group theory and differential equations Birkhäuser, 1993.
28. P. Landesman, Generalized differential Galois theory, *Trans. Amer. Math. Soc.* 360(8), 4441–4495, 2008.
29. B. Malgrange, Sur les déformations isomonodromiques. I. Singularités régulières, *Mathematics and physics (Paris, 1979/1982)*, *Progr. Math.* 37, 401-426, 1983.
30. D. Marker, Model theory of differential fields t Preprint, University of Illinois at Chicago, 2000. (cf. <http://www.msri.org/publications/books/Book39/files/dcf.pdf>)
31. D. Marker, Model theory, algebra and geometry *Math. Sci. Res. Inst. Publ.* 39, 53-63, Cambridge Univ. Press.
32. C. Mitschi, M. F. Singer, Monodromy groups of parameterized linear differential equations with regular singularities *Proc. of the Amer. Math. Soc.* 141, 605-617 (2011).
33. C. Mitschi, M. F. Singer, Projective Isomonodromy and Galois Groups, *Bull. London Math. Soc.* 44 (5), 913-930 (2012).
34. A. Minchenko, A. Ovchinnikov, M. F. Singer, Unipotent differential algebraic groups as parameterized differential Galois groups *Journal of the Institute of Mathematics of Jussieu* 13 (04), 671-700 (2014)
35. A. Minchenko, A. Ovchinnikov, M. F. Singer, Reductive linear differential algebraic groups and the Galois groups of parameterized linear differential equations *International Mathematics Research Notices* (2013), to appear.
36. M. D. Morley, Categoricity in power, *Trans. Amer. Math. Soc.* 114, 514-538, 1965.
37. Y. Ohyaama, Quadratic equations and monodromy evolving deformations *arXiv :0709.4587v1 [math.CA]* 28 Sep 2007.
38. Y. Ohyaama, Monodromy evolving deformations and Halphen's equation in Groups and Symmetries, *CRM Proc. Lecture Notes* 47, Amer.Math.Soc., Providence, RI, 2009.
39. R. S. Palais, Some analogues of Hartogs theorem in an algebraic setting, *Amer. J. Math.* 100(2), 387–405, 1978.
40. M. van der Put and M. F. Singer, Galois Theory of Linear Differential Equations, *Grundlehren der mathematischen Wissenschaften* 328, Springer-Verlag, 2003.
41. A. Robinson, On the concept of differentially closed field, *Bull. Res. Council Israel Sect. F* 8, 113-118, 1959.
42. A. Robinson, Introduction to Model Theory and the Metamathematics of Algebra, *North Holland Publ., Amsterdam*, 1963
43. G. E. Sacks, The differential closure of a differential field, *Bull. Amer.Math. Soc.* 78 (5), 629-634, 1972.
44. R. Schäfke, Formal fundamental solutions of irregular singular differential equations depending on parameters, *J. Dynam. Control Systems* 7 (4), 501–533, 2001.
45. L. Schlesinger, *Handbuch der Theorie der Linearen Differentialgleichungen*, Teubner, Leipzig, 1887.

- 46. A. Seidenberg, Abstract differential algebra and the analytic case, Proc. Amer. Math. Soc. 9, 159-164, 1958.
- 47. A. Seidenberg, Abstract differential algebra and the analytic case II. Proc. Amer. Math. Soc. 23, 689-691, 1969.
- 48. S. Shelah, Differentially closed fields, Israel. J. Math. 25, 314-328, 1976.
- 49. M. F. Singer, Linear algebraic groups as parameterized Picard-Vessiot Galois groups, Preprint, 2011.
- 50. Y. Sibuya, Linear Differential Equations in the Complex Domain: Problems of Analytic Continuation, Translations of Mathematical Monographs, Volume 82 American Mathematical Society, 1990
- 51. C. Tretkoff and M. Tretkoff, Solution of the Inverse Problem in Differential Galois Theory in the Classical Case, Amer. J. Math. 101, 1327-1332, 1979.
- 52. C. Wood, The model theory of differential fields revisited, Israel Journal of Mathematics 25, 1976.
- 53. H. Zoladek, The Monodromy Group, Monografie matematyczne, Inst. Mat. PAN, 67, New Series, Birkhäuser, 2006.

INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE, UNIVERSITÉ DE STRASBOURG, 7 RUE RENÉ DESCARTES,  
67084 STRASBOURG CEDEX, FRANCE

*E-mail address:* mitschi@math.unistra.fr